Jan 10, 2017

Monday, January 9, 2017 2:46 PM

What is a metric, or a metric space?

Either learned in MATH 3060 or refer to Prep-Notes: Metric Spaces

For the moment, let us look at $\mathbb{R}^2$.

Notation: $x=(x_1,x_2)$, $y=(y_1,y_2)$, etc.

The usual distance, or standard metric,

$$\|x-y\| = \left[(x_1-y_1)^2+(x_2-y_2)^2\right]^{1/2}$$

You may have heard of some other

$$d_p(x,y) = \|x-y\|_p$$

$$= \left[|x_1-y_1|^p+|x_2-y_2|^p\right]^{1/p}$$

$$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$$

They are called the ℓ_p-metrics, $p\geq 1$

What are their differences?

More precisely, is there any difference about limit, convergence, continuity, etc., between using different ℓ_p-metrics?

01-10-p2

Tuesday, January 10, 2017 4:59 PM

How do we know that using different ℓ_p will give the same analysis ?

What is the most essential argument ?

Think about limit or convergence, we need

$\forall\, \varepsilon > 0$, ..., $d_p(x_n, x) < \varepsilon$

In the case of continuity, we need

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$, ...,

$d_p(x,y) < \delta$, ..., $d_q(f(x), f(y)) < \varepsilon$

Take $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$ and

$d_\infty(x,y) = \max\{ |x_1 - y_1| , |x_2 - y_2| \}$

Obviously, $d_\infty(x,y) \le d_1(x,y)$

$\therefore d_1 < \varepsilon \Rightarrow d_\infty < \varepsilon$

and $d_1(x,y) \le 2\, d_\infty(x,y)$

$\therefore d_\infty < \varepsilon \Rightarrow d_1 < \frac{\varepsilon}{2}$

Exercise. Assume $p \ge q$. Then

$d_p(x,y) \le d_q(x,y) \quad \forall\, x, y \in \mathbb{R}^2$

$\exists$ fixed constant K (depends on p, q)

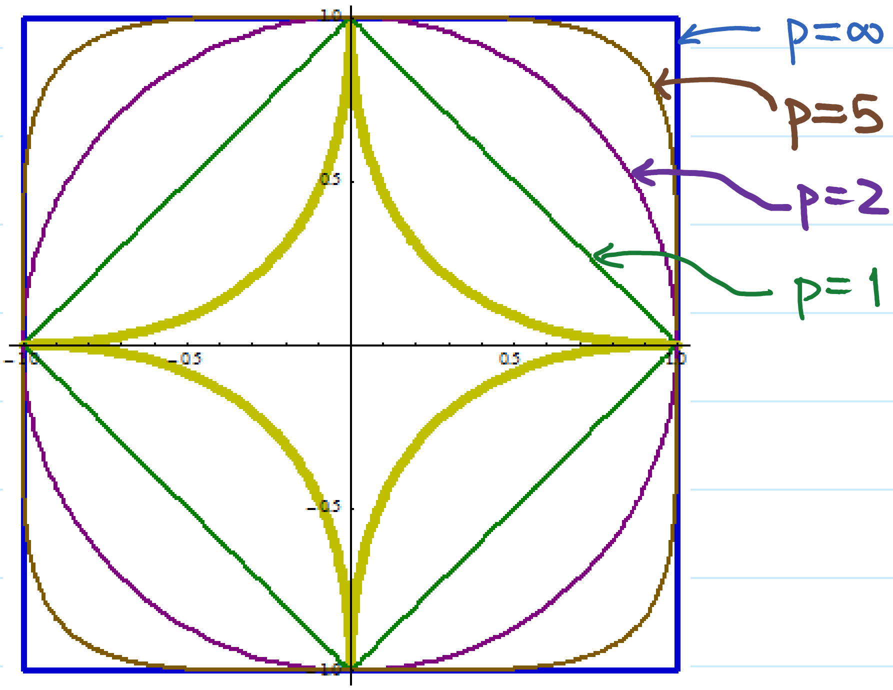
$K\, d_p(x,y) \ge d_q(x,y) \quad \forall\, x, y \in \mathbb{R}^2$

02-10-p3

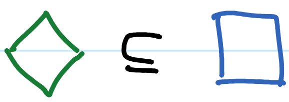
Tuesday, January 10, 2017 5:11 PM

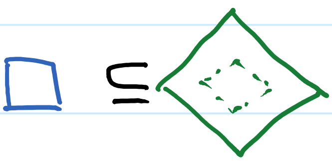
Having seen why all ℓ_p-metric on $\mathbb{R}^n$ behave the same analytically, let us look at them "geometrically"

The pictures below are "circles" of the same radius $r > 0$ for different ℓ_p-metrics



The above inequalities about d_p, d_q can be seen as below.

* $d_\infty \le d_1 \iff$ 

* $d_1 \le 2d_\infty \iff$ 

The same analytical behavior can be described by these sets !!

In the picture, there is a ✧

It is the "circle" defined by $p<1$.
However, for $p<1$, ℓ_p is not a metric.
Exercise Try $p=\frac{1}{2}$, $x=(a,0)$, $y=(0,a)$, $z=(0,0)$
show that $d_p(x,z)+d_p(z,y) < d_p(x,y)$

i.e. The Δ-inequality is not valid.
However, set relations are still true for
✧ ◇ ○ □

We suspect that the same analysis can
be defined even $p<1$ is not a metric
How to define open sets in $\mathbb{R}^2$

① An open set
=
A union of
open balls

② Define interior point
Then $\mathring{A}$ for $A \subset \mathbb{R}^2$
open $\Leftrightarrow$ $A = \mathring{A}$

Clearly, no matter ① or ②, using any
system of ✧ ◇ ○ □ gives the
same open sets

Historically, tried to do analysis without metric

* directly work on convergence
* define neighborhood system
* not tell us what is an open set, but describe $\underbrace{\text{a system}}_{\text{called a topology}}$ containing open sets

When we have such a system, the things inside the system are open sets

Definition. Let X be nonempty. $\mathcal{T} \subset \mathcal{P}(X)$ is a topology if it satisfies

(T1) A union of sets from $\mathcal{T}$ is still in $\mathcal{T}$.

(T2) A finite intersection of sets from $\mathcal{T}$ is still in $\mathcal{T}$.

(T3) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$.

Note. We do not know what is in $\mathcal{T}$, just the rules T1–T3 about $\mathcal{T}$.

Define any $G \in \mathcal{T}$ to be an open set.

Remark. (T1) + (T2) $\Rightarrow$ (T3) by logic

01-10-p6

Tuesday, January 10, 2017 6:09 PM

Mathematically,

(T1) : For each family $\{G_\alpha\}_{\alpha \in I} \subset \mathcal{T}$, we have

$$\bigcup_{\alpha \in I} G_\alpha \in \mathcal{T}$$

Equivalently, $\forall\, \mathcal{G} \subset \mathcal{T}$, $\bigcup \mathcal{G} \in \mathcal{T}$

(T2): For each $\{G_1, G_2, \cdots, G_n\} \subset \mathcal{T}$, we have

$$\bigcap_{k=1}^{n} G_k \in \mathcal{T}$$

Or, $\forall$ finite set $\mathcal{F} \subset \mathcal{T}$, $\bigcap \mathcal{F} \in \mathcal{T}$.

Remark. (T1) and (T2) captures the important experience from $\mathbb{R}^n$.

"Finite" in (T2) is important, as

$$\bigcap_{k=1}^{\infty} \left(-1+\frac{1}{n}, 1-\frac{1}{n}\right) = [-1, 1]$$

Example (1) Metric topology

Given a metric on a set X.

Then one can define open balls

$\left\{ \begin{array}{c} \text{unions of} \\ \text{open balls} \end{array} \right\}$ Interior points

$= \{A = \mathring{A} : A \subset X\}$

Example 2 Discrete Topology arisen from the discrete metric $d(x,y)=\begin{cases} 0 & x=y \\ 1 & x\neq y \end{cases}$

The discrete topology is indeed $\mathcal{P}(X)$.

Reason. For every $x_0 \in X$, $B(x_0,\frac{1}{2})=\{x_0\}$

Thus, if $x_0\in A$ then $x_0 \in \{x_0\} \subset A$ must be an interior point of A

Hence, every A contains each point in its interior, $\therefore A=\mathring{A}$

Example 3 Indiscrete Topology $=\{\phi, X\}$

Example 4 Cofinite Topology

$\mathcal{J}=\{\phi\}\cup\{A\subset X : X\setminus A \text{ is finite}\}$

To verify (T1) and (T2), somehow we use de Morgan's

* $X\setminus \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I}(X\setminus A_\alpha)$

$\subset X\setminus A_\alpha$ is finite

* $X\setminus \bigcap_{k=1}^{n} A_k = \bigcup_{k=1}^{n}(X\setminus A_k)$

finite union of finite sets